HKOI Training

$ami \sim wkc$

Last modified: March 9, 2010

Lecture 02

Propositional Logic and Grammar in C 2 / 38

First Order Propositions 3 / 38

Proposition / Statement

A *statement* / *proposition* is a sentence that has either an answer, "Yes" or "No".[a]
For example, all the following are proposition.[b]

- Today is hot.
- I will not go to school.
- $1 + 2 + 3 = \frac{1}{2}(3)(4)$. (Yes)
- There are infinitely many prime numbers. (Yes)
- $\sqrt{x^2} = x$. (No, it is false when x is negative.)
- If n is a 5-digit square integer, then $n = 29929$. (No)
- $x = 2$ only if $x^2 = 4$. (Yes)
- $x = 2$ if $x^2 = 4$. (No)
- $n = 2$ and n is a prime. (Yes)

4 / 38

[a] We skip a bit by using "common sense" to determine whether a sentence is a proposition or not.
[b] To emphasize that we are not solving equation, we interpret the $=$ sign to be "always equal".

Proposition / Statement

The following are not propositions or we won't discuss the following kind of sentences.

- What time is it now?
- (empty string)[a]
- This statement is false.
- I am lying. [b]
- The second unique child of God is a female.

Actually, some of them can be considered as statements.
However, for simplicity, we shall avoid them at this moment.

5 / 38

[a] This is usually called the ϵ-string
[b] The Liar paradox

Proposition operators

Given some propositions,
we can create new propositions from them by using *logical connectives*.
Be careful, we don't interpret the meaning at this stage.
For example[a],

- NOT(Today is hot).
- NOT(I will not go to school).
- Today is hot AND I will go to school.
- If today is hot, then I will not go to school.
- $x > 3$ OR $x < -1$.
- Every x is greater than 3.
- There is a number which is less than -1 or greater than 3.

6 / 38

[a] We don't care about grammar or tense. What we are interested in the new proposition only.

Negation - NOT

The negation of a proposition P is $\sim P$.
Some book use $\neg P$ to denote the negation.
It is simply a proposition prefixed by a word "not".

- NOT(Today is hot).
- NOT(I will not go to school).
- NOT($x > 3$).
- NOT(x is a prime).

7 / 38

Conjunction - AND

The conjunction of two propositions P, Q is $(P) \wedge (Q)$.
We will denote the conjunction usually by $(P)\, and\, (Q)$ instead.
It connects two propositions by adding by a word "and".
We may sometimes omit the parentheses as well as long as the meaning is clear.

- (Today is hot) AND (I will go to school).
- Today is hot AND I will go to school.
- NOT(I will not go to school) AND NOT(Today is hot).
- ($x > 2$) AND (x is even).

8 / 38

Disjunction - OR

The disjunction of two propositions P, Q is $(P) \vee (Q)$.
We will denote the disjunction usually by $(P)\, or\, (Q)$ instead.
It connects two propositions by adding by a word "or".
We may sometimes omit the parentheses as well as long as the meaning is clear.

- (Today is hot) OR (I will go to school).
- Today is hot OR I will go to school.
- NOT(I will not go to school) OR NOT(Today is hot).

9 / 38

Implication - IF-THEN

The implication of two propositions P, Q is "IF (P) THEN (Q)".
We will denote the implication usually by "$P \implies Q$" instead.
It connects two propositions by adding by an arrow or using the words "if" and "then".
We may sometimes omit the parentheses as well as long as the meaning is clear.

- IF (Today is hot) THEN (I will go to school).
- IF (($x > 2$) AND (x is even)) THEN (NOT(x is a prime)).
- NOT(I will not go to school) OR NOT(Today is hot).

Definition. *Let P and Q be propositions,*

- *The* **converse** *of an implication $P \implies Q$ is $Q \implies P$.*[a]
- *The* **inverse** *of an implication $P \implies Q$ is $\sim P \implies \sim Q$.*
- *The* **contrapositive** *of an implication $P \implies Q$ is $\sim Q \implies \sim P$.*

10 / 38

[a] It is sometimes denoted by $P \impliedby Q$.

Biconditional - IF-AND-ONLY-IF

The bi-conditional of two propositions P, Q is "(P) IF AND ONLY IF (Q)".
We will denote the bi-conditional usually by "$P \iff Q$" or "P iff Q" instead.
It connects two propositions by adding by an bi-arrow.

- ($ax^2 + bx + c = 0$ has solution) IF AND ONLY IF ($b^2 - 4ac \geq 0$) .
- (n is a composite) IF AND ONLY IF (NOT(n is prime)).
- (Two lines are parallel) IF AND ONLY IF (NOT(They meet at a point)).

We don't interpret the correctness of the above proposition, this is discussed in next section.
Indeed, if you consider the correctness, not all of them are always true.

11 / 38

Truth value

Truth value are a value, either "*false*" or "*true*", associated to each proposition.
The association of the value must obey the following laws:

1. Each proposition has ONE value each time.

2. Every propositions constructed by propositional operators will have a corresponding value.

Definition. *A proposition that is always having the truth value "true" is called a* tautology*.*
A proposition that is always having the truth value "false" is called a contradiction*.*

To demonstrate the first law,
for example, using our common sense to associate the truth value to the following:

- $1 + 1$ is equal to $2.$ (Tautology)

- The Earth is a square. (Contradiction)

- Today is hot. (Just a proposition)

Boolean operations - NOT

Suppose P is a proposition and it has a truth value.
Are the truth value of P and $\sim P$ related?

The second law state that they are related according to some rules, which is given as follow.

P	$\sim P$
true	false
false	true

That means, whenever P is associated with a value "true" , $\sim P$ must have the value "false".
And whenever P is associated with a value "false" , $\sim P$ must have the value "true".

Boolean operations - AND

Suppose P and Q are propositions and have truth value.
Similarly, the truth value of "P and Q" are related by the following table.

P	Q	P and Q
true	true	true
true	false	false
false	true	false
false	false	false

To interpret the table, it is equal to ask whether both propositions are true.

15 / 38

Boolean operations - OR

Suppose P and Q are propositions and have truth value.
Similarly, the truth value of "P or Q" are related by the following table.

P	Q	P or Q
true	true	true
true	false	true
false	true	true
false	false	false

To interpret the table, it is equal to ask whether at least one of the propositions is true.

16 / 38

Boolean operations - IF-THEN

Suppose P and Q are propositions and have truth value.
Similarly, the truth value of "$P \implies Q$" are related by the following table.

P	Q	$P \implies Q$
true	true	true
true	false	false
false	true	true
false	false	true

It is difficult at first to accept this table.

For example,
the proposition "IF (The earth is a square) THEN ($1 + 1 = 3$)" has a value "true".
The correct interpretation is that
"whether one can determine the statement is honest or not and if so, is it honest?"
One can determine a people is lying only when the condition holds,
otherwise we can say that is a joke rather than a lie.

17 / 38

Boolean operations - IF-AND-ONLY-IF

Suppose P and Q are propositions and have truth value.
The truth value of "$P \iff Q$" are related by the following table.

P	Q	$P \iff Q$
true	true	true
true	false	false
false	true	false
false	false	true

The bi-condition is true only when both propositions have the same truth value.

Definition. *Let P and Q be two propositions,*
P and Q are **logically equivalent** *if $P \iff Q$.*

The equivalence is in a sense that
by merely looking at the truth value of two propositions, we cannot distinguish them.
So, that means the two propositions are logically the same.

Theorem. *Let P_1 and P_2 be two propositions,*
and $P_1 :=$"$P \iff Q$", $P_2 :=$"$(P \implies Q)$ and $(Q \implies Q)$".
P_1 and P_2 are logically equivalent.

18 / 38

Boolean Algebra

Let P, Q and R be propositions, $\mathcal{T}$ be a tautology and $\mathcal{F}$ be a contradiction.

Prove that the following pairs are equivalent:

$\sim \mathcal{T}$	$\mathcal{F}$
$\sim \mathcal{F}$	$\mathcal{T}$
$\sim\ \sim P$	P
P and $\sim P$	$\mathcal{F}$
P or $\sim P$	$\mathcal{T}$
$\sim (P$ and $Q)$	$\sim P$ or $\sim Q$
$\sim (P$ or $Q)$	$\sim P$ and $\sim Q$
(P and Q) and (R)	(P) and (Q and R)
(P or Q) or (R)	(P) or (Q or R)
(P and Q) or (R)	(P or R) and (Q or R)
(P or Q) and (R)	(P and R) or (Q and R)
(P or Q) and (P)	P
(P and Q) or (P)	P

Example

Let Q and R be propositions,

P_1 be the proposition that "$Q \implies R$",

P_2 be the proposition that "not Q or R".

P_3 be the proposition that "(not R) $\implies$ (not Q)"

Proof. We first show that $P_1 \iff P_2$ is true by computing all cases.

Q	R	$\sim Q$	$Q \implies R$	$\sim Q$ or R	$P_1 \iff P_2$
true	true	false	true	true	true
true	false	false	false	false	true
false	true	true	true	true	true
false	false	true	true	true	true

Next, we show that $P_2 \iff P_3$ as follow:

$P_3 = \sim R \implies \sim Q \iff \sim (\sim R)$ or $\sim Q$

the proposition $\sim (\sim R)$ or$Q \iff R$ or $\sim Q = P_2$ □

Example

Let P be the proposition that "n is a five-digit square integer whose digits are all 2 and 9",
Q be the proposition that "n is 29929."
The above two are equivalent.

Proof. Show that $P \implies Q$ and $Q \implies P$.
$Q \implies P$: Check that $29929 = 173^2$.
$P \implies Q$: Read lecture 1. □

Second Order Propositions

Propositional variables

A proposition may be depend on variable(s).
For example, we let $P(n)$ be the proposition that "n is a prime number.".
Then we have infinitely many propositions depends on n, say

- $P(6)$ is the proposition "6 is a prime number" .
- $P(11)$ is the proposition "11 is a prime number" .
- $P(123)$ is the proposition "123 is a prime number" .
- ...

Let $Q(x, y)$ be the proposition that "x is smaller than y" [a]
For example, $Q(\text{John}, \text{Mary})$ is the proposition that "John is smaller than Mary".
$Q(2, 3)$ is the proposition that "2 is smaller than 3".

[a] The values of a variable need not be a number.

Existential quantifier - THERE EXISTS

As like what we did for those first order logical connectives,
we can construct new proposition by using second order logical connectives.
Let $P(n)$ be the proposition that "n is a prime number.".
We can create a new proposition that "There is an integer k such that $P(k)$".
It is denoted by "$\exists k(P(k))$".
However, to avoid so many parentheses, it is usually denoted as "$\exists k,\ P(k)$".
Its truth values depends on all the proposition $P(n)$,
it is true if there is at least one proposition having the value "true".
Since $P(2)$ is true, "$\exists k,\ P(k)$" is true.
Simply because there is such an integer.

Universial quantifier - FOR ALL

Let $P(n)$ be the proposition that "n is a prime number.".
We can create a new proposition that "Every integer k such that $P(k)$".
It is denoted by "$\forall k(P(k))$".
However, to avoid so many parentheses, it is usually denoted as "$\forall k,\ P(k)$".
Its truth values depends on all the proposition $P(n)$,
it is true if all propositions are having the value "true".
As $P(4)$ is false, "$\forall k,\ P(k)$" is false.
Simply because not all of them are "true".

Theorem. *The following are true:*
"not($\forall x, P(x)$*)* $\iff \exists x,$ *not*$P(x)$*"*[a]
"not($\exists x, P(x)$*)* $\iff \forall x,$ *not*$P(x)$*"*[b]

Question:
Can we interchange the operators FOR ALL and THERE EXISTS?
i.e. Are the two proposition "$\forall x, \exists y, Q(x, y)$" , "$\exists y, \forall x, Q(x, y)$" the same?
Hints: Let $Q(x, y)$ be the proposition that "x is smaller than y".
"$\forall x, \exists y, Q(x, y)$" means
for every number x, there is an number y such that $x < y$.
i.e. For each given number, there is a larger number.
"$\exists y, \forall x, Q(x, y)$" means
there is an number y such that every number x is smaller than y.

[a] Not every proposition are true means that at least one of them is false.
[b] Not having at least one true means that all of them are false.

Quick introduction

a **recursive definition** (or **inductive definition**) is used to define an object in terms of itself[a].
For example,
let a_n be numbers defined as follows:

1. $a_0 = 1$
2. $a_n = n \cdot a_{n-1}$, for $n > 0$

To find a_6, we put $n = 6$ and use the second one, $a_6 = 6 \cdot a_5$ and so on...
until we arrive at a_0 which is a known value and we therefore know $a_6 = 720$.

Definition. *The* n**-th factorial**, $n!$*, is defined as the value of* a_n *as above.*

Recursive definition is something like above as long as the objects involved are well-defined.

[a] P. Aczel (1977), "An introduction to inductive definitions", Handbook of Mathematical Logic, J. Barwise (ed.)

Polynomial evaluation

Let f be a polynomial and $f(x) = 3x^5 + 2x^3 + 5x - 7$.
To find $f(2)$, we usually compute $3 \cdot 2^5, 2 \cdot x^3, \cdots$ and add them together.
Here is a better method for find the value $f(2)$.

Define a_n be the coefficient of x^n of $f(x)$,
i.e. $a_0 = -7, a_1 = 5, a_2 = 0, a_3 = 2, a_4 = 0, a_5 = 3$
let x_n be numbers defined as follows:

1. $x_0 = a_5$
2. $x_n = 2 \cdot x_{n-1} + a_{5-n}$, for $5 \geq n > 0$

The value of $f(2)$ is exactly x_5.
What are the other numbers $x_0, \cdots, x_4$ used for?
Try to divide $f(x)$ by $(x - 2)$ using polynomial long division.
If we let $b_n = x_{4-n}$ for $0 \leq n \leq 4$, then the polynomial $Q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4$ is exactly the quotient.
Recursive definition is something like about as long as the objects involved are well-defined.

Polynomial evaluation

$$
\begin{array}{r|l}
 & 3x^4 + 6x^3 + 14x^2 + 28x + 61 \\
\hline
x - 2 & 3x^5 \quad + 2x^3 \quad + 5x - 7 \\
 & -3x^5 + 6x^4 \\
\hline
 & 6x^4 + 2x^3 \\
 & -6x^4 + 12x^3 \\
\hline
 & 14x^3 \\
 & -14x^3 + 28x^2 \\
\hline
 & 28x^2 + 5x \\
 & -28x^2 + 56x \\
\hline
 & 61x - 7 \\
 & -61x + 122 \\
\hline
 & 115
\end{array}
$$

Synthetic substitution

Read the Extra Material

31 / 38

Grammar in C

32 / 38

English Grammar

For those who study linguistic, they view the grammar of English systematically.

- Each passage consists of a title, author and a sequence of paragraphs
- Each paragraph is a sequence of sentences
- Every sentence end with a full-stop (.) or an exclamation mark (!).
- <sentence> ::= <subject> <verb> [<object>] [a]
- Every subject is a noun-phrase, verb-phrase etc.
- Every noun-pharse are consists of a smaller noun-phrase, noun, relative pronoun etc.
- Every noun are vocabulary.
- Every vocabulary is consisting of correctly spelled character sequences.
- A character sequence is consists of characters
- Each character is from ’a’ to ’z’ or ’A’ to ’Z’

33 / 38

[a] Every sentence consists of a subject and a verb, object is optional

C Grammar - BNF Grammar

An analogue linguistic structure for programming language C also exists.

- Each program consists of pre-processor directives and functions[a]
- Each function consists of statements.
- Every statements ends with a semi-colon (;).
- Every statement is either a control flow , variable declaration or expression.
- expression can be arithmetic expression, logical expression etc.
- arithmetic expression consists consists of addition, subtraction etc.
- <addition>::=<Arithmetic Operand 1> '+' <Arithmetic Operand 2>
- Arithmetic Operand can be function values, variables names or numbers.
- numbers can be integers or real numbers.
- <integers>::= [0-9]+, i.e. at least one of any one of '0','1', $\cdots$, '9'

34 / 38

[a] A part to tell the computer to store action.

Arithmetic Expression

We learnt the way to convert usual mathematical expression into C language.
However, how could we tell the computer to do the following two different action?

$3 \div 2$	Quotient: 1
$3 \div 2$	Real division: 1.5

Even ourselves cannot distinguish the two different division without further explanation.
Computer will use the following rules to distinguish the two different division.

1. If every operands are computer-integers, it perform the quotient division.[a]
2. If any one of the operand is not a computer-integer, it will switch to real division.

Computer-integers means that the number is “written in a form”
so that the computer treat it as an integer.
For example, we can say 2.3 is not a whole number due to the decimal place.

35 / 38

[a] Computer like doing discrete mathematics too!

Regular Expression

To specify clear what we mean by "written in a form", we need **Regular Expression**.
It is a tool used for string matching.
For example,

[0-9]	matches any one of the character '0','1',$\cdots$,'9'
[0-9]+	matches any string that is consists of at least one digit.
a*bc	matches any string that is end with "bc" and start with any number of "a".
[^a-z]	matches any character that is not one of 'a','b',$\cdots$,'z'.
ak4|bb10	matches any one of the string "ak4" or "bb10".
(ab)|(kaab)	matches any one of the string "ab" or "kaab"[a].
(a|b)+	matches any string that is consists of 'a' and 'b' and is non-empty.
a|b+	matches any string "a" , "b" , "bb", "bbb" , $\cdots$.
(\+|\-)?	matches any string "+" , "-" or ϵ-string , $\cdots$.

36 / 38

[a] The parentheses here is used for grouping

Computer Integers and Real numbers

Although we won't distinguish usually between 2.5 and 3, there are just number.
Computer would like to do so.
It only knows whole number.

Computer integer is of the form (\+|\-)?[0-9]+
Therefore, it will treat $3/2$ to be 1.

For simplicity, we denote [DIGIT] to be [0-9] and [INT] to be [DIGIT]+
Computer treats other numbers to be computer-real number, it can be "-2.3" , "2.0", "1E+12"[a].
Let [SIGN] $::=$ (\+|\-),
Computer real number is of the form [SIGN]?[DIGIT]*"."[DIGIT]+((e|E)[SIGN]?[INT]) or [SIGN]?[DIGIT]+(e|E)[SIGN]?[INT].

Therefore, it will treat $3/2.0$ to be 1.5 , $3.0/2$ to be 1.5 and $3.0/2.0$ to be 1.5.

37 / 38

[a] Scientific notation 1×10^{12}.