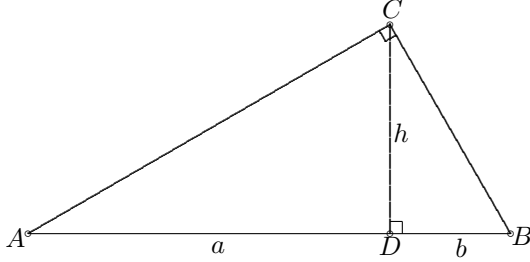


WEEK 1 EXERCISE SOLUTION

Problem 1. In the following diagram, $AD = a$, $DB = b$ and $CD = h$.



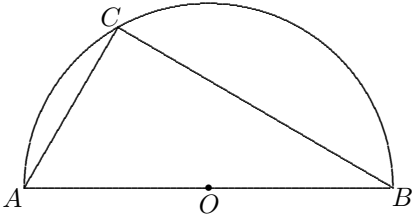
Show that $\triangle ACD \sim \triangle CDB$ and $h = \sqrt{ab}$

Proof. Let $\angle DCB = x$, we have $\angle DCA = 90^\circ - x = \angle DBC$ and $\angle DAC = x$

Hence $\triangle ACD \sim \triangle CDB$ (AA)

$$\frac{CD}{DB} = \frac{AC}{CD} \iff CD^2 = AC \cdot DB \iff h^2 = ab \iff h = \sqrt{ab} \quad \square$$

Problem 2. In the following diagram, AB is the diameter, C is a point on the semi-circle and O is the center.



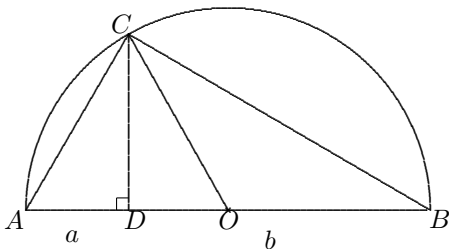
Show that $\angle ACB = 90^\circ$.

Proof. Draw OC and let $\angle OCA = x$, $\angle OCB = y$, so $\angle ACB = x + y$.

Since $OC = OA = OB$, we have $\angle OAC = x$ and $\angle OBC = y$.

Therefore, $2x + 2y = 180^\circ \iff x + y = 90^\circ \quad \square$

Problem 3. Point D is the foot of the perpendicular from C to AB and $AD = a$ and $DB = b$.



From the previous questions, we have $\angle ACB = 90^\circ$ and hence $DC = \sqrt{ab}$.

(1) Find OC and OD in terms of a and b .

Proof. $OC = \frac{a+b}{2}$ and $OD = OA - DA = \frac{a+b}{2} - a = \frac{b-a}{2} \quad \square$

Theorem (Pythagoras). Let $\triangle ODC$ be with sides $OD = a, DC = b, OC = c$.

$\angle ODC$ is a right triangle if and only if $a^2 + b^2 = c^2$.

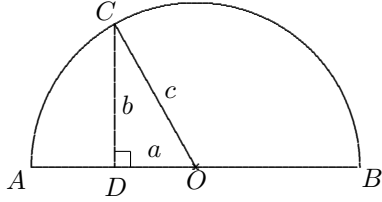
Proof. We proved the \implies direction first.

For each given right triangle $\triangle ODC$ with sides a, b and c .

Draw a circle with center O and OC as the radius.

Extend the line OD to get the intersection points A and B .

Consider only the semi-circle which contains the triangle $\triangle ODC$.



Then $AD = c - a$ and $DB = AB - DA = 2c - (c - a) = c + a$.

Problem 2 implies that $\angle ACB$ is a right angle.

Hence from problem 1, we have $DC = \sqrt{DA \cdot DB} = \sqrt{(c - a)(c + a)} = \sqrt{c^2 - a^2}$

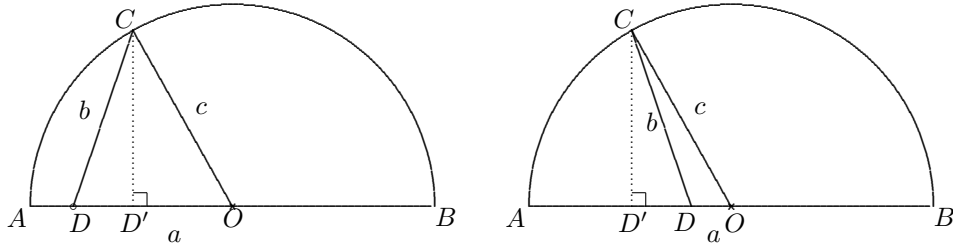
On the other hand, $DC = b$. Hence $b = \sqrt{c^2 - a^2} \implies a^2 + b^2 = c^2$.

Conversely, \Leftarrow), if $a^2 + b^2 = c^2$, assume $\angle ODC$ is not a right angle, it can either be an acute or obtuse angle.

We still proceed by drawing the semi-circle which contains the triangle and D still lies on AB .

Here, we use the **method of false position**. Assume D is at a false position.

Let D' be the foot of perpendicular of C to AB , which is the true position.



We shall prove that D and D' must be the same point by considering $d = DD'$ be their distance.

If $\angle ODC$ is acute, $OD' = OD - DD' = a - d$. Otherwise, $OD' = OD + DD' = a + d$

Since $\triangle OD'C$ is a right triangle, we have $CD'^2 = OC^2 - OD'^2$.

$$\therefore CD'^2 = c^2 - (a \pm d)^2 = c^2 - a^2 \mp 2ad - d^2 = b^2 \mp 2ad - d^2$$

On the other hand, $\triangle CD'D$ is also a right triangle, hence $CD'^2 + DD'^2 = CD^2$.

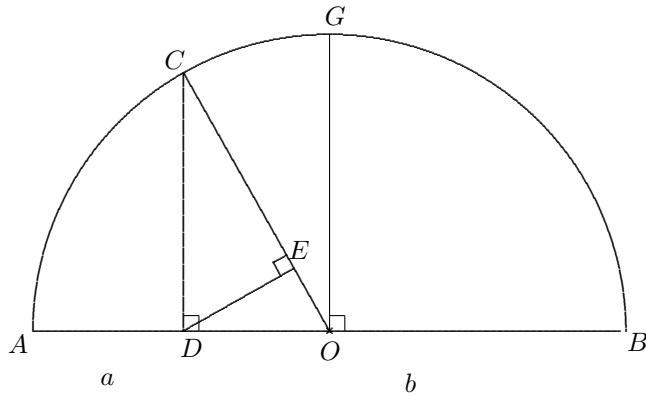
$$\therefore (b^2 \mp 2ad - d^2) + (d^2) = b^2 \iff \mp 2ad = 0 \implies d = 0, \text{ since } a > 0.$$

Therefore, $DD' = 0$ which means they must be the same point.

Hence, $\triangle ODC = \triangle OD'C$ is a right triangle. □

Problem 4 (Construction of Pythagorean mean). In the diagram below, $AD = a$ and $DB = b$.

G is the intersection of the the perpendicular bisector of AB and the semi-circle.



- (1) Show that $\triangle CDO \sim \triangle CED$ and $CE = \frac{2ab}{a+b} = \frac{2}{\frac{1}{a} + \frac{1}{b}}$.
- (2) Draw the line segment DG and show that $DG = \sqrt{\frac{a^2+b^2}{2}}$.

Proof. Using the same argument as in problem 1, the two triangles are similar.

$$\text{Also, } \frac{CE}{CD} = \frac{CD}{CO} \iff CE = \frac{CD^2}{CO} = \frac{1}{OC} \cdot CD^2 = \frac{2}{a+b} (\sqrt{ab})^2 = \frac{2ab}{a+b} = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

$$OD = \frac{b-a}{2} \text{ and } OG = \frac{b+a}{2}.$$

$$\text{Using the pythagoras theorem, } DG^2 = OD^2 + OG^2 = \frac{a^2+b^2}{2}. \quad \square$$

C'S SYNTAX

$$(1) \quad x^3 + y^3 = x*x*x+y*y*y$$

$$(2) \quad (x+y)^2(x-y) = (x+y)*(x+y)*(x-y)$$

$$(3) \quad \sqrt{(x-y)^2} = \text{sqrt}((x-y)*(x-y))$$

$$(4) \quad (\sqrt{x-y})^2 = \text{sqrt}(x-y)*\text{sqrt}(x-y)$$

$$(5) \quad a^2 + 2ab + b^2 = a*a+2*a*b+b*b$$

$$(6) \quad \frac{x^3 + y^3}{x+y} = (x*x*x+y*y*y)/(x+y)$$

$$(7) \quad x^2 - xy + y^2 = x*x - x*y + y*y$$

$$(8) \quad \left(\frac{x^3 + y^3}{x+y}\right)^2 = (x*x*x+y*y*y)/(x+y) * (x*x*x+y*y*y)/(x+y)$$