Chapter 1 Logarithms

How to solve the following equations?
$10^x = 1000$. Obviously, $1000 = 10^3$, $\therefore x = 3$
How about this?
$4^x = 8$. By observation, $4^x = 2^{2x} = 2^3$, $\therefore x = 1.5$
But this is not as easy as the previous problem.
How about $10^x = 30$?
What is the value of $10^{1.4}$? $10^{1.5}$? $10^{1.45}$? $10^{1.475}$? $10^{1.477}$?
Is it getting closer and closer to 30?
With this observation, we believe that there is some real number x such that $10^x = 30$.
Such number x is denoted by $\log x$.
In fact, the number x must exist and must be unique.

Definition of common logarithm:
A positive real number y is equal to $\log x$ if and only if $x = 10^y$.
This is called the **logarithm with base 10** or **common logarithm**.

You may check $\log 30$ *with your calculator to see if the answer is closed to 1.477*

Therefore, we have to following formulas.
(1) $\log(ab) = \log a + \log b$
(2) $\log\left(\frac{a}{b}\right) = \log a - \log b$
(3) $\log 1 = 0$
(4) $\log 10 = 1$
(5) $a \log b = \log(b^a)$

Why are the above formulas true??
To show (1), we notice that $x = \log a$, $y = \log b$ means $a = 10^x$, $b = 10^y$.
Also, $z = \log(ab)$ means $ab = 10^z$, from above, we also have $ab = (10^x)(10^y) = 10^{x+y}$.
$\therefore z = x + y$. So, $\log(ab) = \log a + \log b$.
To show (2), replace a with $\frac{a}{b}$ in (1).
(3) and (4) are left as self-practice.
* *(5) is too difficult to show unless you have knowledge of pure math in F6.* *

Logarithm of base other than 10:
A positive real number y is equal to $\log_a x$ if and only if $x = a^y$ for $a > 0$ and $a \neq 1$.
Does (1) to (5) still hold when using other bases?

All holds but (4), in fact, we always have $\log_a a = 1$.

Notes prepared by Wu, Kai Chiu
Email: ami.wu[at]family.ust.hk
Course homepage: http://game.integate.net/hkoi/2008/

Important formula between logarithm and power:

In the last chapter, there are 2 questions.
We have already answered the first one.
For the second one, we can use the formula to calculate x^y.

$$x^y = 10^{y \log x}$$

Change of bases:

Mathematically, we can write down $\log_{13} 10$ easily.
But, in our calculator, we usually only have the $\log x$ function only.
So, how can we find the decimal value of $\log_{13} 10$?

We can find the value by this formula $\log_a b = \dfrac{\log_c b}{\log_c a}$.

Examples:

$\log 100 = \log(10 \cdot 10) = \log 10 + \log 10 = 1 + 1 = 2$

$\log 0.1 = \log \dfrac{1}{10} = \log 1 - \log 10 = 0 - 1 = -1$

$\log_2 32 = \log_2 2^5 = 5 \log_2 2 = 5$

$\log_{13} 10 = \dfrac{\log 10}{\log 13} = \dfrac{1}{\log 13}$

$\log 2 + \log_{100} 2 = \log 2 + \dfrac{\log 2}{\log 100} = \log 2 + \dfrac{\log 2}{2} = \dfrac{3 \log 2}{2}$

Application of common logarithm:

The number of digits of any positive integer can be known with the help of $\log$.
Observation:
$\log 100 = 2$, $\log 1000 = 3$, $\log 500 \approx 2.7$
So, it seems like that all number $100 \le x < 1000$ will have $2 \le \log x < 3$.
The **lower bound** 2 is exactly less than the number of digit of x by one.
No. of digits = $\lfloor \log x \rfloor + 1$.

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Chapter 1 Exercises

(You are not allowed to use calculator)

Find the value of the following:

1) $\frac{\log_3 81}{\log_3 \sqrt{3}}$

2) $\log_2 2^{31}$

3) $10^{\log 12345}$

4) $7^{\log 20} \times 7^{\log 5}$

5) Simplify $\frac{\log \sqrt{x}}{2\log x + \log x^5}$, where $x > 0$ and $x \neq 1$

6) If $a = \log 2$, $b = \log 5$, simplify the following in terms of a, b.

(a) $\log 20$

(b) $\log \frac{1}{8}$

(c) $\log 0.2$

(d) $\log 0.4$

(e) $\log \frac{1}{20}$

7) Show that $a \log b = \log(b^a)$ is true when a is a positive integer.

8) Solve the equation $\log x + \log(x-7) = -10$

End of Chapter